Math 363 Homework Due Monday, November 20

Consider the problem of modeling the spread of an infectious disease in a population (such as the flu on a college campus). We make the following assumptions about the disease:

- Only a few people have the disease to begin with, and a larger population is capable of catching the disease from them.
- A person who has contracted and recovered from the disease is then immune to the disease.
- The disease has a negligibly short incubation period.
- The total population is constant. (This assumption means that we ignore births, deaths from natural causes, and immigration.)
- The population is divided into three classes of people: the infective class I, the susceptible class S, and the removed class R. The infective class consists of people who have the disease and are capable of transmitting the disease to others. The susceptible class consists of people who have not yet had the disease but are capable of catching the disease and becoming infective. The removed class consists of people who had the disease and died, or had the disease and recovered and are now immune, or have the disease and are isolated.
- The rate of change of the susceptible population S is proportional to the product of the number of members of S and the number of members of the infective class I. (The number of contacts between two groups is often assumed to be proportional to the product of the group sizes.) Question: What becomes of susceptible people who are no longer susceptible?
- People are removed from the infective class I at a rate proportional to the size of I.

Let $S(t)$ and $I(t)$ denote the numbers of people in classes S and I, respectively, at time t. (We ignore the removed class R in what follows.) A mathematical model for the spread of the disease then consists of the system of two first-order differential equations

$$\frac{dS}{dt} = -aSI\,,$$
$$\frac{dI}{dt} = aSI - bI\,,$$

where a and b are positive constants. The constant a is the *infection rate*; it measures how fast the disease is transmitted from the infected population to the susceptible population. The constant b is the *removal rate*; it represents the rate at which the infected population enters the removed population (that is, dies, recovers and is immune, or is isolated). In what follows, we take $a = 0.001$ and $b = 0.75$.

(continued)

1. Use your model to determine what happens if initially only one person is infected and 2500 persons are susceptible (Gettysburg College's student population). A phase plane analysis of the system in the first quadrant gives all the information you need.
   a. Does the disease die out rapidly or does an epidemic occur? You have to decide what an epidemic is.
   b. What is the largest number of infected people at any time?
   c. Do any susceptibles remain after the disease dies out?
2. Suppose the people in the susceptible class S are vaccinated against the disease at a rate proportional to their number with proportionality constant 0.25.
   a. How is the model changed?
   b. What happens now? (Use the same initial conditions as in #1.) Does an epidemic still occur?
   c. Do any susceptibles remain after the disease dies out?