First Player's Cannot-Lose Strategies for Cylinder-Infinite-Connect-Four with Widths 2 and 6

Yoshiaki Yamaguchi[1](✉) and Todd W. Neller[2]

[1] The University of Tokyo, Tokyo, Japan
yamaguchi@klee.c.u-tokyo.ac.jp
[2] Gettysburg College, Gettysburg, USA

Abstract. Cylinder-Infinite-Connect-Four is Connect-Four played on a cylindrical square grid board with infinite row height and columns that cycle about its width. In previous work, the first player's cannot-lose strategies have been discovered for all widths except 2 and 6, and the second player's cannot-lose strategies have been discovered with all widths except 6 and 11. In this paper, we show the first player's cannot-lose strategies for widths 2 and 6.

1 Introduction

We begin by introducing the two-player game of Cylinder-Infinite-Connect-Four. We call the first and second players *Black* and *White*, respectively. Cylinder-Infinite-Connect-Four is played on a square grid board that wraps about a semi-infinite cylinder (Fig. 1). Rows extend infinitely upward from the ground, and we number columns of a width w board with indices that cycle rightward from 0 to $w - 1$ and back to 0. Players alternate in dropping disks of their colors to the lowest unoccupied grid cell of each drop column. Thus a game *position*, i.e. a configuration of disks, is unambiguously described as a sequence of column numbers. For clarity, we additionally prefix each column number with the first letter of the player color, so "Bi" or "Wi" means that Black or White, respectively, places a disk in column i.

The object of the game is to be the first player to place four or more of one's own disks in an adjacent line horizontally, vertically, or diagonally. We call such a four-in-a-row a *Connect4*. Because of the cylindrical nature of the board, the Connect4 is further constrained to consist of 4 different disks. Thus, a horizontal Connect4 is not allowed for widths less than 4. If, for a given state and given player strategies, we can show the impossibility of either player ever achieving a Connect4, the value of the game is said to be *draw*.

We call a configuration of disks a position. When a player places a disk, background of the cell is colored gray. We duplicate columns 0 through 2 to the right on wider boards to allow easy inspection of wraparound Connect4 possibilities. Figure 2 shows an example terminal game position after B0W0B2W2B1W3B5. A *threat* is defined as a single grid cell that would complete a Connect4 [3].

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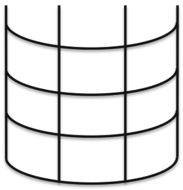

Fig. 1. Board of Cylinder-Infinite-Connect-Four

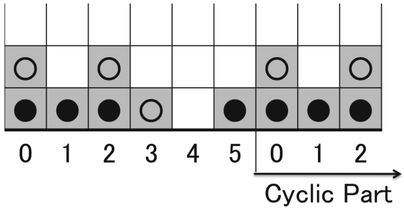

Fig. 2. Example position

After B0W0B2W2B1, Black has a double threat on the bottom row. Although W3 removes one threat, Black can play the other threat B5 and complete a Connect4.

In previous work [1], the first player's cannot-lose strategies have been discovered for all widths except 2 and 6, and the second player's cannot-lose strategies have been discovered for all widths except 6 and 11. In this paper, we show the first player's cannot-lose strategies for widths 2 and 6.

2 Related Work

In 1988, James Dow Allen proved that Connect-Four played on the standard board with width 7 and height 6 is a first player win [2]; 15 days later, Victor Allis independently proved the same result [4]. Results for Connect-Four games played on finite boards with non-standard heights and/or widths were reported in [9]. Yamaguchi et al. proved that Connect-Four played on a board infinite in height, width, or both, leads to a draw by demonstrating cannot-lose strategies for both players [10]. These cannot-lose strategies are based on paving similar to that used in polyomino achievement games [13–16,18] and 8 (or more) in a row [17].

Other solved connection games include Connect6 for special openings [20], the Hexagonal Polyomino Achievement game for some hexagonal polyominoes [18, 19], Gomoku [11], Renju [12], Qubic [5–7], and Rubik's Cube [8]. Other games with cyclic topology include Cylinder Go [21], Torus Go [21], and TetroSpin [22].

3 First Player's Cannot-Lose Strategy for Cylinder-Infinite-Connect-Four for Width 2

In this section, we show the Black cannot-lose strategy for width 2. First, we define a *follow-up* play as a play in the same column where the opponent just played [4]. Figure 3 shows a Black follow-up play. A *follow-up strategy* is a strategy consisting of follow-up plays.

After Black's first play in Cylinder-Infinite-Connect-Four for width 2, each player has only 2 play choices: follow-up or non-follow-up. Black's cannot-lose strategy is summarized as follows:

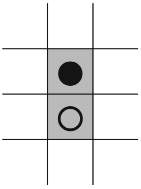

Fig. 3. Black follow-up play

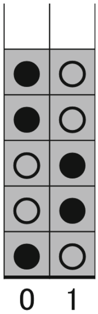

Fig. 4. White never plays follow-up

- As long as White plays a non-follow-up strategy, Black alternates between follow-up and non-follow-up plays, starting with follow-up.
- If White plays a follow-up after Black plays a follow-up (Fig. 5), then Black always plays a follow-up strategy thereafter.
- If instead White plays a follow-up after Black plays the initial move or a non-follow-up (Case 2 of Fig. 6), then Black always plays a follow-up except after White plays a non-follow-up after White plays a follow-up at first.

We now consider this strategy in detail. As long as White does not play follow-up, Black's alternating follow-up and non-follow-up play leads to the game sequence B0W1B1W0B1W0B0W1 (Fig. 4) and the resulting pattern permits no Connect4 for either player. If this play pattern continues, the game is a draw. Thus we now need only to consider the ramifications of Black's response to a White follow-up play.

As soon as White makes a follow-up play, there are 2 cases shown in Figs. 5 and 6 which include mirror-symmetric cases as well. These cases capture both of the essentially different situations that may arise in a White non-follow-up play sequence of any length, including 0. When White plays follow-up in White's first and second moves, the "ground" line Figs. 5 and 6 are in Fig. 4 play sequence.

When White plays follow-up at Case 1, Black responds with follow-up thereafter (Fig. 7). If White were to play in column 1, Black's follow-up response would then win, so White must then continue an infinite follow-up sequence in column 0 to draw.

After Case 2, Black plays follow-up (Fig. 8). In Fig. 8, the bold line within figures serves to highlight pieces below that must have been played. If White only plays a follow-up strategy, the game is drawn. However, if White plays non-follow-up in column 0, then Black plays non-follow-up in column 1 and then plays a pure follow-up strategy thereafter. As can be seen in Fig. 8, Black's lowest diagonal Connect4 *undercuts* White's lowest diagonal Connect4 (highlighted with a zig-zag line), so any efforts of White to complete a Connect4 will result in Black completing a Connect4 first.

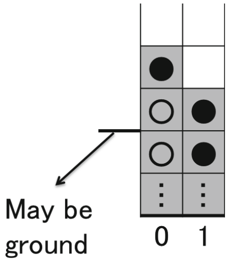

Fig. 5. Case 1

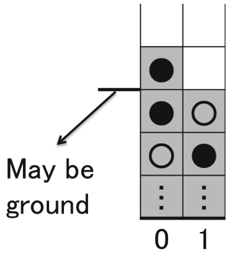

Fig. 6. Case 2

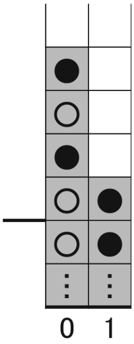

Fig. 7. After Case1

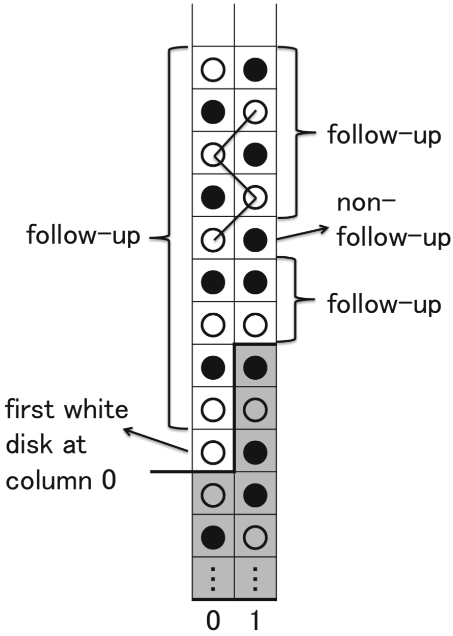

Fig. 8. After Case2

4 First Player's Cannot-Lose Strategy for Cylinder-Infinite-Connect-Four for Width 6

In this section, we show the Black first player's cannot-lose strategy for width 6 via a branching game-tree case analysis. For each possible line of White play, we show that Black can prevent a White Connect4.

B0W1B2- Fig. 9.
B0W{2 or 3}B0- Fig. 10.
B0W0B2W1B0- Fig. 9.
B0W0B2W3B3W1B0- Fig. 9.
B0W0B2W3B3W{2, 4, or 5}B3- Fig. 11.

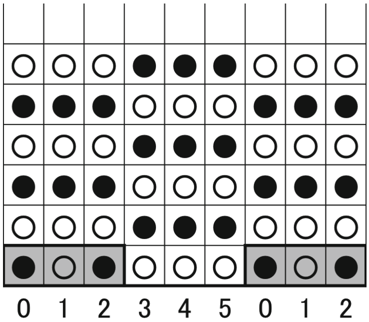

Fig. 9. There is neither Connect4.

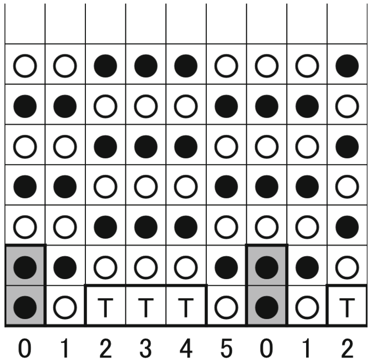

Fig. 10. There is neither Connect4.

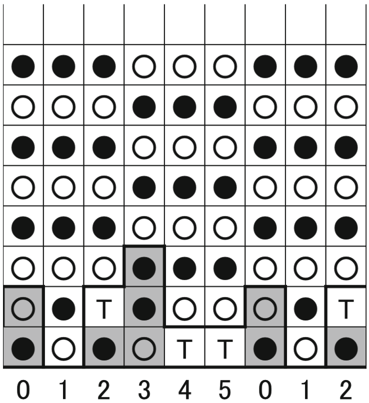

Fig. 11. There is neither Connect4.

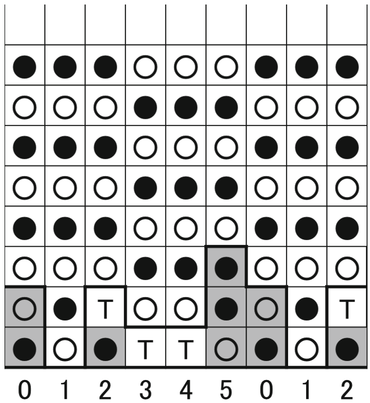

Fig. 12. There is neither Connect4.

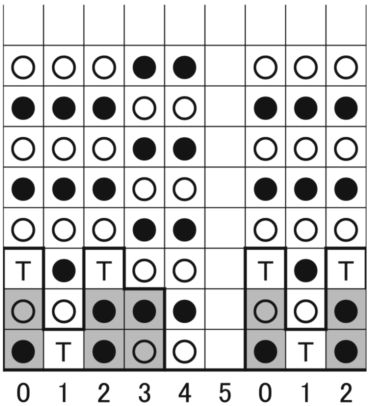

Fig. 13. There is neither Connect4.

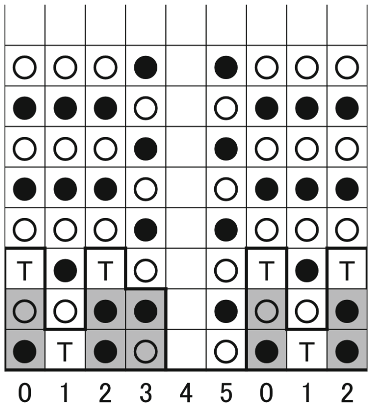

Fig. 14. There is neither Connect4.

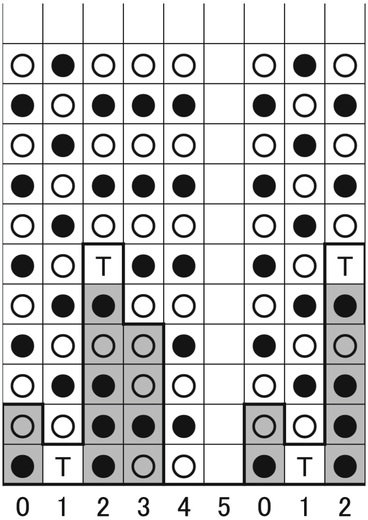

Fig. 15. There is neither Connect4.

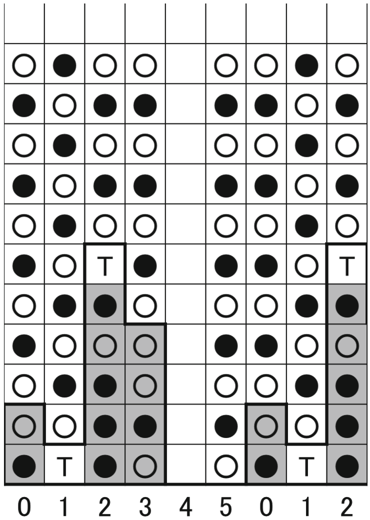

Fig. 16. There is neither Connect4.

B0W0B2W3B3W0B2- Figs. 13 or 14.
B0W0B2W3B3W3B2W{0, 1, or 2}B3- Figs. 13 or 14.
B0W0B2W3B3W3B2W3B2W2B2- Figs. 15 or 16.
B0W0B2W5B5W1B0- Fig. 9.
B0W0B2W5B5W{2, 3, or 4}B5- Fig. 12.
B0W0B2W5B5W0B2- Figs. 17 or 18.
B0W0B2W5B5W5B2W{0, 1, or 2}B5- Figs. 17 or 18.
B0W0B2W5B5W5B2W5B0- Figs. 19 or 20.

We begin our explanation of this case analysis by observing that after Black's play in column 0, we may ignore symmetric board positions and only treat the cases where White plays in columns 0 through 3. When B0W1, Black plays in the column 2 and then plays only follow-up afterward (Fig. 9). Above the highest bold line in all board figures of this section, Black's disk is always on White's disc. Some pieces above the bold line may have been played. All possible 4 × 4 subboards of each board are present so that all possible Connect4 achievements may be visually checked.

After B0W2B0 or B0W3B0, we mark 3 cells with a "T" as in Fig. 10 and note that White's disk occupies one of these 3 T cells. When White plays in one of the 2 remaining empty T cells, Black immediately responds by playing in the other. This pattern of play is repeated in other figures with T cells.

If White does not reply in the column 1, 3, or 5 after B0W0B2, then Black can create a bottom-row double threat (as in Fig. 2) by playing in column 1 and achieve a Connect4 on the next turn in column 3 or 5. Black's follow-up strategy response to B0W0B2W1 is shown in Fig. 9.

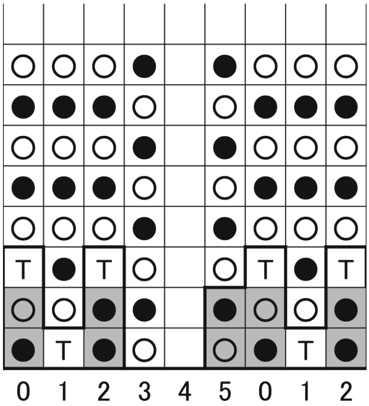

Fig. 17. There is neither Connect4.

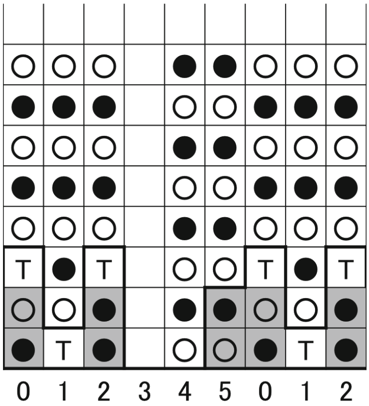

Fig. 18. There is neither Connect4.

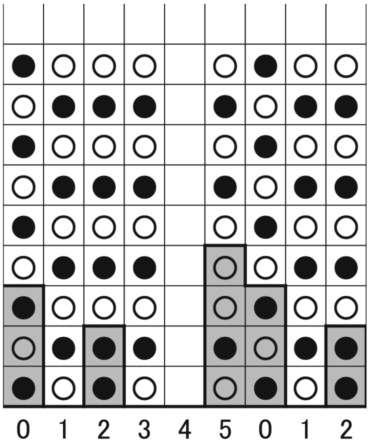

Fig. 19. There is neither Connect4.

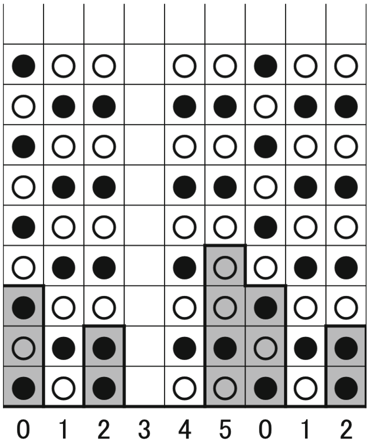

Fig. 20. There is neither Connect4.

After B0W0B2W3B3, White can play in any of the columns 0–5. We show cases B0W0B2W3B3W1B0 and B0W0B2W3B3W{2, 4, or 5}B3 in Figs. 9 and 11, respectively.

After B0W0B2W3B3W0 and B0W0B2W3B3W3, Black plays in column 2 and establishes row 2 threat with playing only follow-up in columns 4 and 5. Because of this 2 threats, White cannot place a disk in both columns 4 and 5. Case B0W0B2W3B3W0B2 is shown in Figs. 13 or 14 where Black plays only follow-up in columns 4 and 5. White can play in any of the columns 0–3 except for columns 4 and 5 after B0W0B2W3B3W3. Cases B0W0B2W3B3W3B2W{0, 1, or 2}B3 are shown in Figs. 13 or 14. Case B0W0B2W3B3W3B2W3B2W2B2 are shown in Figs. 15 or 16 where Black plays only follow-up in columns 4 and 5.

B0W0B2W5 cases are similar in nature to those of B0W0B2W3. Note that columns 3 or 4 often fulfill the same lowest-threat role that columns 4 or 5 did in prior cases.

5 Conclusion

In this paper, we have shown the first player's cannot-lose strategies for widths 2 and 6. For width 2, we have shown that a simple pattern prevents White Connect4 for non-follow-up White play, and that the first follow-up White play allows Black to draw with follow-up strategy after a single non-follow-up response. (This is only for Case 2. Case 1 is only follow-up play. And Black may win by some White response.)

For width 6, we have shown a detailed case analysis where two techniques proved most useful: (1) marking "T" positions to establish a shape for a follow-up draw, and (2) establishing a lowest threat to prevent White from playing in 2 columns and then focusing on (1) to force a draw elsewhere.

We conjecture that the same techniques used for first-player width 6 cannot-lose strategy will also be useful in future work for establishing the same result for second-player width 6 and 11 cannot-lose strategies.

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