

Optimal Defensive Strategies in One-Dimensional *RISK*

DARREN GLASS

Gettysburg College
300 N. Washington St.
Gettysburg, PA 17325
dglass@gettysburg.edu

TODD NELLER

Dept of Computer Science
Gettysburg College
300 N. Washington St.
Gettysburg, PA 17325
tneller@gettysburg.edu

RISK is a popular board game invented by Albert Lamorisse and released in 1957. The board depicts a stylized political world map divided into territories, each occupied by one or more of a player's army units, which we will refer to simply as armies. The bulk of play consists of a turn-based series of attacks between player armies occupying adjacent territories in an effort to occupy the entire world. In the late game, it is common for one player to attempt to eliminate another player along a chain of territories.

In this note, we consider the problem of how a defensive player should distribute his armies to maximize the probability of survival. In particular, we will consider a one-dimensional version of the game, which takes place on a chain of $m + 1$ consecutive territories, as depicted in FIGURE 1. We now describe the rules of our version of the game. Experienced *RISK* players will note that this is a significantly simplified version of the game, but we believe that we have captured most of the spirit of the original game. At the end of the paper, we will discuss some ways in which the differences in the actual game of *RISK* might affect our proposed strategies.

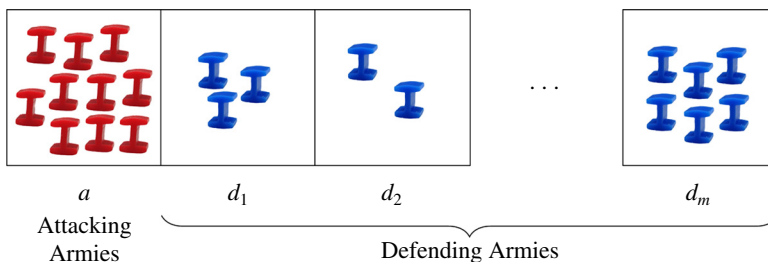


Figure 1 One-dimensional *RISK* board

At the beginning of the game we assume that the attacker has a positive number of armies (labeled a in FIGURE 1) on a territory at one end and the defender has a number of armies (labeled d_1, \dots, d_m) distributed among the other m territories so that there is at least one army per territory. The goal of the attacker is to take over all m of the defender's territories while the goal of the defender is to prevent this from happening.

The attacker begins by rolling a number of dice that is the lesser of 3 and the number of attacking armies minus 1. The defender then simultaneously rolls a number of dice that is the lesser of 2 and the number of defending armies. (We note that in the actual game of *RISK* each player may choose to roll a smaller number of dice than allowed. We assume that they always choose to roll the maximum number allowed.)

The dice rolled by attacker and defender are then sorted and compared. The maximum attacking die roll is compared to the maximum defending die roll. If a second attacker–defender die pair was rolled, the second-to-maximum die rolls are compared as well. For each attacker–defender die pair, the attacker wins if and only if the attacking die roll is greater than the defending die roll, i.e., the maximum roll wins with ties going to the defender. For each loss, the attacker or defender removes a single army from the relevant territory. Thus, the result of a roll of three attacking dice versus two attacking dice can result $(a_{\text{loss}}, d_{\text{loss}}) \in \{(0, 2), (1, 1), (2, 0)\}$, where a_{loss} and d_{loss} are the number of armies lost for the attacker and defender, respectively.

As long as the attacker has two or more armies remaining and the defender has any armies remaining, we assume that the attacker will repeat the attack roll process. If the attacker drops below two armies then we declare the defending army the winner. If the defending armies are successfully eliminated, the territory is captured, and the attacker must occupy the territory. We assume that the attacker leaves precisely one army behind on her territory and moves the remaining armies onto the defender’s territory. Assuming she has moved at least two armies, she continues to attack the next territory in the defender’s chain. The process repeats until the attacking player either drops below two armies on her leading territory or she captures all of the territories in the defending player’s chain.

This version of the game captures the essence of the situation in the actual game of *RISK* in which players have large numbers of army reinforcements (i.e., from occupation of entire continents) or are playing *RISK* variations where one can obtain a large number of armies by trading in card sets. In those situations, the maximum number of armies is often moved into a territory after one successful capture. The attacker then immediately seeks to capture a territory adjacent to the captured territory. This process often has the goal to occupy a continent or possibly eliminate an opponent from the game. It is thus of great interest to understand how the defender army distribution affects the probability of defender survival.

In the spirit of two earlier articles in *THIS MAGAZINE*, we implement a Markov chain model of the game of *RISK*. In [4], Tan develops such a model in order to answer the question of when it is worthwhile for a player to attack an adjacent territory and what the expected damage to the attacker in such a battle will be. Several years later, Osborne found that Tan’s model made overly strong assumptions of independence, and in [2], he corrects those assumptions and addresses additional questions of strategy under his corrected Markov chain model. However, he continues to only look at the strategy in situations where one territory attacks another, while we look at the more general question of how to distribute armies among multiple territories in order to optimize one’s chances of survival. Looking at the numerical results of this model, as seen below, we have formulated the following conjecture.

Conjecture. *In order to best survive an attack on a chain of m territories with d armies where $d \geq 2m$ and the number of attacking armies is sufficiently large, the best defense is to place two armies on each of the first $m - 1$ territories and the remaining armies on the last territory.*

Pieces of this conjecture have been found in the folklore of the game of *RISK*, and in particular it is asserted without justification in [1] that one wants to defend with an even number of armies whenever possible. We note that this conjecture is not true if

the number of attacking armies is small: In particular, we will show that a different strategy is optimal if the attacker only has $m + 1$ armies, which is the smallest number that can be used in a campaign against m territories with any hope of success. While we are unable to prove the full conjecture, we will prove the following theorem.

Theorem 1. *In order to best survive an attack on a chain of m territories with n armies, the best defense will be to place either one or two armies on each of the first $m - 1$ territories and the remaining armies on the last territory.*

We will also consider a variant on our question in which it is assumed that the attacker has an overwhelmingly large number of armies and is therefore always rolling three dice. Under this assumption, we will consider how the defender should place her armies so as to cause the most damage to the attacker, even though he will eventually lose all of his armies. While this question is not quite the same situation as we would like to consider in the game of *RISK*, it shares many similarities and the fact that the best strategy is the same as in the conjecture gives us some evidence that the conjecture is correct. A final section discusses other considerations in the actual gameplay of *RISK* that may change the way a player would apply our results in practice.

Our numerical experiments and theoretical computations rely on the following probabilities, computed by Osborne in [2]. Suppose that for a given battle the defending player has one army on her territory and the attacking player rolls three dice. It is an elementary (if tedious) probability calculation to see that the probability that the value of the defender's roll is at least as high as the highest of the attacker's roll is $\frac{441}{1296}$, and we denote this probability by \tilde{p}_0 . Similarly, the probability that the defender loses the one army in this battle is $\frac{855}{1296}$, which we denote by \tilde{p}_1 . Osborne further computes the probabilities of all possible outcomes of a given set of die rolls, which we give in FIGURE 2. In our notation, the letter is dependent on the number of dice rolled by the attacker, with p meaning three dice, q meaning two dice, and r meaning a single die. The presence or lack of a tilde depends on whether the defender is rolling one or two dice, respectively, and the subscript is the number of armies lost by the defender.

Att. Dice	Def. Dice	Def. Loss	Att. Loss	Prob.	Notation
3	2	2	0	$\frac{2890}{7776}$	p_2
3	2	1	1	$\frac{2611}{7776}$	p_1
3	2	0	2	$\frac{2275}{7776}$	p_0
3	1	1	0	$\frac{855}{1296}$	\tilde{p}_1
3	1	0	1	$\frac{441}{1296}$	\tilde{p}_0
2	2	2	0	$\frac{295}{1296}$	q_2
2	2	1	1	$\frac{420}{1296}$	q_1
2	2	0	2	$\frac{581}{1296}$	q_0
2	1	1	0	$\frac{125}{216}$	\tilde{q}_1
2	1	0	1	$\frac{91}{216}$	\tilde{q}_0
1	2	1	0	$\frac{55}{216}$	r_1
1	2	0	1	$\frac{161}{216}$	r_0
1	1	1	0	$\frac{15}{36}$	\tilde{r}_1
1	1	0	1	$\frac{21}{36}$	\tilde{r}_0

Figure 2 Probabilities for different outcomes based on the number of dice rolled

Empirical observations

In this section, we use a Markov chain model of the game of *RISK* in order to gather some data that we then use to formulate our conjecture. Recall that our version of *RISK* assumes that the attacker armies begin amassed on a single territory adjacent to a chain of m territories occupied by the defending player. We begin by simulating the battle between the attacker and the defender in the first territory in the defender's chain. If the attacker wins this battle, we assume that she maximally occupies the territory, leaving behind a single occupying army. She proceeds to attack the defender's second territory with the remaining armies, continuing as she wins each additional territory.

We have performed brute-force computations to gain a sense of what distribution of the defender's armies will lead to the highest probability that he survives the full attack. For a given number of total armies, we consider all nontrivial divisions between the number of attacker armies a and defender armies d . In this context, "nontrivial" means that the defender has more than one possible distribution to consider, and the optimal probability for survival is neither 0 nor 1. For each distribution of defending armies, we computed the probability that the defending player had armies remaining at the end of the battle.

To get a flavor of these experiments, consider the case where an attacker with 30 armies seeks to eliminate a defender with 30 armies distributed along an adjacent chain of five defender territories. If we distribute the defending armies uniformly along the first four chain territories and place the remainder on the last chain territory, we can see that all even-numbered uniform army distributions yield greater survival probabilities than each odd-numbered uniform army distribution, as computed from our Markov chain model and described in FIGURE 3.

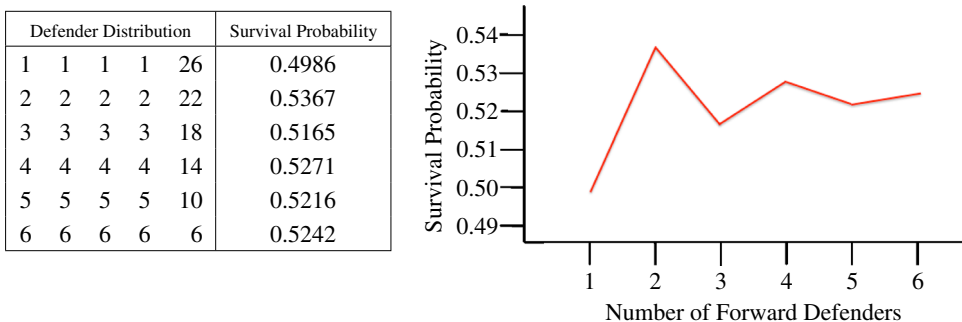


Figure 3 Scenarios where defender distributes armies uniformly

Given a number of defending armies, we consider each length m of defender territory chains that lead to legal, nontrivial distributions. Each territory must be occupied, so this amounts to the condition $2 \leq m \leq d - 1$. Given a , d , and m , we then consider all possible distributions of defender armies, compute the survival probability of each distribution, and test hypotheses on the optimal distributions. We note that when d is close to m , then the defender does not have much flexibility in distributing the armies, and in this situation we will say that the defender is "highly constrained." Similarly, when a is close to m , we will say that the attacker is "highly constrained."

The results of this experiment are contained in FIGURE 4 with axes a , d , and m . Each glyph represents a class of optimal distribution for a given a , d , and m . In considering all possible scenarios up to 46 total armies, we observe several patterns that were the basis for our conjecture and theorems.

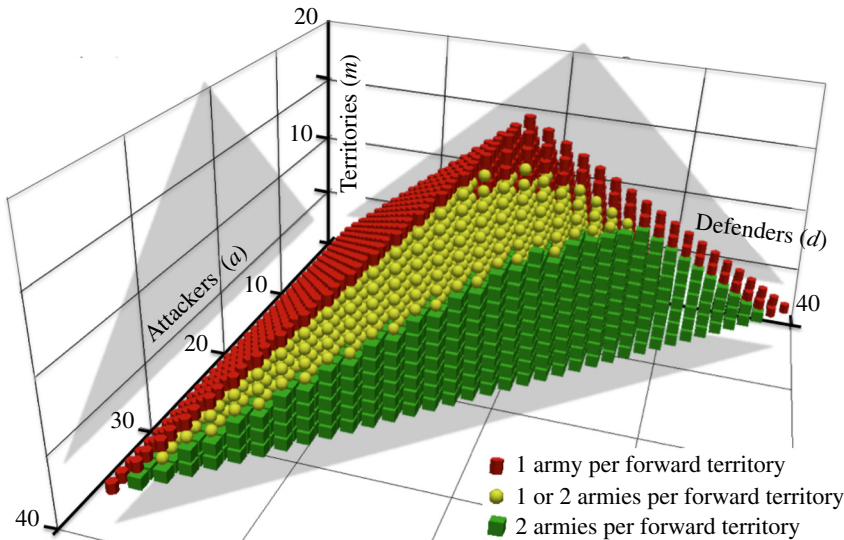


Figure 4 Optimal distributions

- In all cases, the optimal strategy consists of placing either one or two armies on all but the final territory. We prove that this is the case as Theorem 3.
- When $a \leq m + 2$, the defender's optimal army distribution has a minimal forward defense of one army per territory with all remaining armies on the last territory. We call this the *one-army strategy* and Theorem 2 shows that it is optimal in the case where $a = m + 1$.
- When the attacker has a sufficiently large force, the defender's optimal army distribution has a forward defense of two armies per territory with all remaining armies on the last territory, as long as the defender has enough armies to use this distribution (i.e., $d \geq 2m$). We refer to this as the *two-army strategy* and while we have not been able to find sharp bounds for what we mean by "sufficiently large," the number appears to be bounded below by the total number of territories plus a linear function of the number of attacker armies.

To see situations in the "in-between" cases where the defender uses a mix of the one- and two-army defense, we let $e = a - m$ be the number of attacker armies in excess of the number of defender territories. For each $4 \leq e \leq 10$, FIGURE 5 shows the smallest number of total armies for which an optimal distribution calls for fewer than two armies per forward space when it is possible to defend with two armies per forward space. In each case, note that the number of attacking armies is slightly less than half the total number of armies.

We observe that one-army distributions occur only in situations in which the defender is highly constrained. Usually, either the attacker must have just enough armies to occupy all territories given perfect attacking rolls, or the defender has one or two armies to distribute beyond the minimum one-army per space for occupation. Exceptions occur when the number of attacking armies and defending armies are nearly equal and both players are highly constrained. All cases of optimal one-army distributions are highly constrained for at least one of the players.

We next observe that in most other cases where the defender can defend with two armies per forward territory, it is optimal to do so. Most mixed distribution cases are out of necessity; the defender hasn't enough armies to defend with two per territory.

Defender Territories (m)	Attacker Armies (a)	Defender Armies (d)	$e = a - m$	Optimal Distribution
4	8	9	4	2, 1, 2, 4
5	10	11	5	2, 1, 2, 2, 4
6	12	13	6	2, 2, 1, 2, 2, 4
8	15	17	7	2, 2, 2, 2, 1, 2, 2, 4
9	17	19	8	1, 2, 2, 2, 2, 2, 2, 2, 4
10	19	21	9	2, 1, 2, 2, 2, 2, 2, 2, 2, 4
11	21	23	10	2, 2, 1, 2, 2, 2, 2, 2, 2, 2, 4

Figure 5 “Mixed strategy” distributions that are optimal

We also note that the division between mixed and two-army distributions lies almost exactly along the plane defined by $d = 2m$, with exceptions occurring only in cases where the attacker is highly constrained. Most importantly, we note that if neither player is highly constrained, and the defender can defend with a two-army distribution, it is usually optimal to do so.

These observations led us to make the conjecture in the introduction, and related results are discussed in the final sections of this article. FIGURES 6, 7, and 8 depict several cross sections of the three-dimensional array in FIGURE 4 that we found helpful in understanding the situations in which various strategies were optimal.

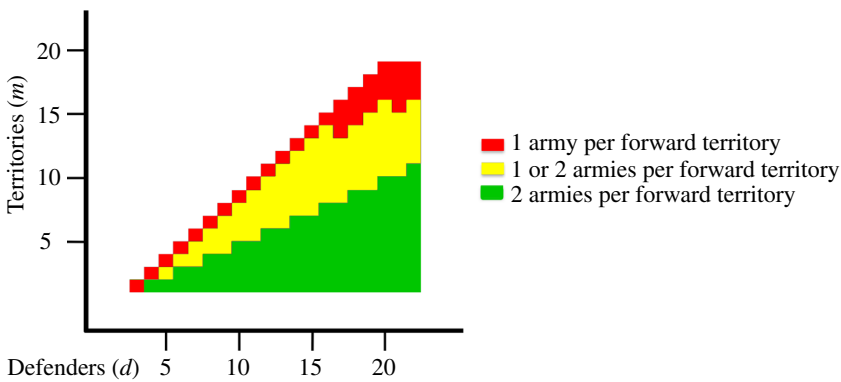


Figure 6 Cross section where $a = 20$

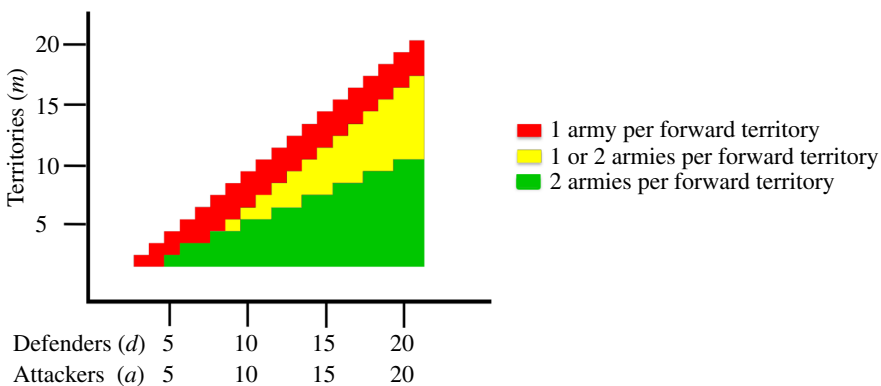


Figure 7 Cross section where $a = d$

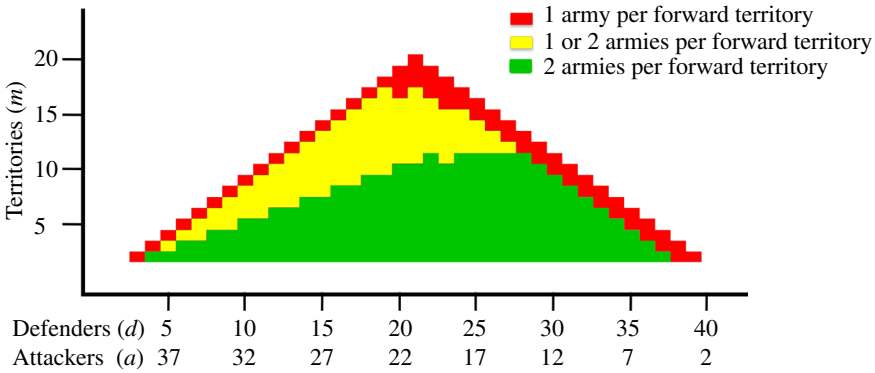


Figure 8 Cross section where $a + d = 42$

Holding off a limited number of attacking armies

In this section, we consider strategies based on a fixed number a of attacking armies and a chain of length m compared to a .

Defending $a - 1$ territories We begin by considering the case where $m = a - 1$ and therefore the attacker is highly constrained. In particular, in this case, we are able to prove the following theorem giving an explicit optimal strategy.

Theorem 2. *In order to best survive an attack on a chain of m territories where the number of attacking armies is $m + 1$, the best defense is to place one army on each of the first $m - 1$ territories and the remaining armies on the last territory.*

We note that if the attacker is successful, then she will need all $m + 1$ of her armies in order to occupy the $m + 1$ total spaces involved. In particular, the defending player will be successful in defending his chain if and only if he manages to defeat a single attacking army during the campaign.

If $m \geq 3$, then in the battle over the first territory, the attacking army will have at least four armies and therefore will be able to roll the full three dice. In particular, the probability that the attacker will defeat k armies without losing a single one of her own armies is given by $p_2^{\lfloor \frac{k}{2} \rfloor} \tilde{p}_1^{\bar{k}}$ where $\bar{k} = 0$ if k is even and $\bar{k} = 1$ if k is odd. This same formula will hold in each of the first $m - 2$ territories. For the battle over the $(m - 1)$ st territory, the formula is similar, only now the attacker is only allowed to roll two dice, so it becomes $q_2^{\lfloor \frac{k}{2} \rfloor} \tilde{q}_1^{\bar{k}}$. For the final territory, the attacker is only allowed to roll one die, and therefore the probability she is successful in a battle against k armies is $r_1^{k-1} \tilde{r}_1$.

Putting this all together, we see that if the defending player distributes his d armies so that there are d_i armies on the i th territory for $i = 1, \dots, m - 1$ and the remaining $d_m = d - \sum d_i$ armies on the final territory, then the probability that the defender will lose all of his territories is given by the following expression:

$$F(d_1, d_2, \dots, d_{m-1}) = p_2^{\lfloor \frac{d_1}{2} \rfloor} \tilde{p}_1^{\overline{d_1}} \dots p_2^{\lfloor \frac{d_{m-2}}{2} \rfloor} \tilde{p}_1^{\overline{d_{m-2}}} q_2^{\lfloor \frac{d_{m-1}}{2} \rfloor} \tilde{q}_1^{\overline{d_{m-1}}} r_1^{d - \sum d_i - 1} \tilde{r}_1.$$

In particular, we note that increasing d_1 by 1 leads to the following identity:

$$F(d_1 + 1, d_2, \dots, d_{m-1}) = \begin{cases} \frac{p_2}{\tilde{p}_1 r_1} F(d_1, d_2, \dots, d_{m-1}) \approx 2.21 F(d_1, d_2, \dots, d_{m-1}) & \text{if } d_1 \text{ odd} \\ \frac{\tilde{p}_1}{r_1} F(d_1, d_2, \dots, d_{m-1}) \approx 2.59 F(d_1, d_2, \dots, d_{m-1}) & \text{if } d_1 \text{ even.} \end{cases}$$

Increasing d_1 by one will always lead to decreasing the defender’s probability of success, and therefore the defender should choose d_1 to be as small as possible. Specifically, he should place a single army on the first territory. The exact same computation holds for territories 2 through $m - 2$. To consider the $(m - 1)$ st territory, we see from our formula that

$$F(d_1, d_2, \dots, d_{m-1} + 1) = \begin{cases} \frac{q_2}{\tilde{q}_1 r_1} F(d_1, d_2, \dots, d_{m-1}) \approx 1.54 F(d_1, d_2, \dots, d_{m-1}) & \text{if } m \text{ is odd} \\ \frac{\tilde{q}_1}{r_1} F(d_1, d_2, \dots, d_{m-1}) \approx 2.27 F(d_1, d_2, \dots, d_{m-1}) & \text{if } m \text{ is even,} \end{cases}$$

and again we see that the defender should choose d_{m-1} to be as small as possible.

Defending $a - 2$ territories In situations in which the attacker has more armies and can withstand losses, the formula to compute the probability that the attacker wins is not quite as simple as in the previous theorem and in particular breaks down into different cases depending on how many armies the attacker loses and when she loses these armies. In particular, when attacking a territory that defended by k armies, then one can see that there are $\lceil \frac{k}{2} \rceil$ ways in which the attacker can lose one army. Therefore, if the defending army spreads its d armies with d_1 armies on the first territory, d_2 on the second, etc., then there are a total of $\lceil \frac{d_1}{2} \rceil + \dots + \lceil \frac{d_m}{2} \rceil$ ways in which the campaign can play out with the attacker losing a single army and defeating all m territories. We note that this sum depends only on the parities of the d_i . In particular, if $d_1 \geq 3$, then we note that there are the same number of cases to consider if we instead distribute the armies with $d_1 - 2$ armies on the first territory, $d_m + 2$ armies on the final territory, and d_i armies on the i th territory for all other i . Moreover, these cases pair up in a natural way depending on when the attacker loses the single army—in one case, she loses it on the first roll of the dice, in another the second roll, etc. Depending on when this loss occurs with respect to the breaks between territories, the probability of a given case occurring might be different, as we see in the following example.

Example 1. We consider two scenarios in which the defending army has eight armies to defend four territories against six attacking armies. In Scenario A, the defender splits the armies 3/2/2/1 and in Scenario B he splits the armies 1/2/2/3. In each scenario, there are five battles in which the attacker might lose a single army and therefore the defender wins overall. The following table gives the probability of each of these cases for each scenario, as well as the case where the attacker does not lose any armies, along with the relative attacker advantage given by Scenario A.

Battle lost	Scenario A	Scenario B	Attacker Advantage for A
None	$p_2 \tilde{p}_1 p_2 p_2 \tilde{q}_1$	$\tilde{p}_1 p_2 p_2 q_2 \tilde{q}_1$	$\frac{p_2}{q_2}$
First	$p_1 p_2 p_2 q_2 \tilde{r}_1$	$\tilde{p}_0 \tilde{p}_1 p_2 q_2 r_1^2 \tilde{r}_1$	$\frac{p_1 p_2}{\tilde{p}_0 \tilde{p}_1 r_1^2}$
Second	$p_2 \tilde{p}_0 \tilde{p}_1 p_2 q_2 \tilde{r}_1$	$\tilde{p}_1 p_1 \tilde{p}_1 q_2 r_1^2 \tilde{r}_1$	$\frac{p_2^2 \tilde{p}_0}{p_1 \tilde{p}_1 r_1^2}$
Third	$p_2 \tilde{p}_1 p_1 \tilde{p}_1 q_2 \tilde{r}_1$	$\tilde{p}_1 p_2 p_1 \tilde{q}_1 r_1^2 \tilde{r}_1$	$\frac{\tilde{p}_1 q_2}{\tilde{q}_1 r_1^2}$
Fourth	$p_2 \tilde{p}_1 p_2 p_1 \tilde{q}_1 r_1$	$\tilde{p}_1 p_2 p_2 q_1 r_1 \tilde{r}_1$	$\frac{p_1 \tilde{q}_1}{q_1 r_1}$
Fifth	$p_2 \tilde{p}_1 p_2 p_2 \tilde{q}_0 \tilde{r}_1$	$\tilde{p}_1 p_2 p_2 q_2 \tilde{q}_0 \tilde{r}_1$	$\frac{p_2}{q_2}$

In all six cases, one can compute that the attacker’s advantage is greater than one in Scenario A. In particular, because the probability of the attacker winning the campaign is the sum of these six cases, each of which prefers the attacker in Scenario A, it is clear that the defending player should prefer Scenario B and therefore divide the armies as 1/2/2/3.

More generally, one can consider what the various possibilities are for the change in the attacker’s probability of winning when two armies are shifted from the last territory to the first territory if one fixes the battle in which the attacker suffers her only casualty. In the following table, we consider all of the possibilities of where this loss can occur as well as the parity of the number of defending armies in this territory as well as the previous territory and the impact that will be felt if the defender moves these two armies.

Territory of Loss	Battle of Loss Within Terr.	Parity of Defenders in Terr. of Loss	Parity of Defenders in Prior Terr.	Advantage for Attacker	Numerical Advantage
No loss				$\frac{p_2}{q_2}$	1.63
Final	Not First	Any	Any	$\frac{p_2}{q_2}$	1.63
Final	First	Any	Even	$\frac{p_1 \tilde{q}_1}{q_1 r_1}$	2.35
Final	First	Any	Odd	$\frac{p_2 \tilde{p}_0 \tilde{q}_1}{p_1 q_1 r_1}$	1.34
Penultimate	Last	Any	Any	$\frac{p_1 q_2}{p_0 p_1 r_1^2}$	5.33
Penultimate	Middle	Any	Any	$\frac{q_2}{r_1^2}$	3.5
Penultimate	First	Even	Even	$\frac{\tilde{p}_1 q_2}{\tilde{q}_1 r_1^2}$	3.93
Penultimate	First	Even	Odd	$\frac{p_2 \tilde{p}_0 q_2}{p_1 \tilde{q}_1 r_1^2}$	2.28
Penultimate	First	Odd	Even	$\frac{\tilde{p}_1 \tilde{q}_1}{r_1^2}$	5.78
Penultimate	First	Odd	Odd	$\frac{p_2 \tilde{p}_0 \tilde{q}_1}{p_1 r_1^2}$	3.35
Other	Last	Any	Any	$\frac{p_1 p_2}{p_0 p_1 r_1^2}$	8.69
Other	Middle	Any	Any	$\frac{p_2}{r_1^2}$	5.72
Other	First	Even	Even	$\frac{p_2}{r_1^2}$	5.72
Other	First	Odd	Even	$\frac{\tilde{p}_1^2}{r_1^2}$	6.49
Other	First	Even	Odd	$\frac{p_2^2 \tilde{p}_0}{p_1 \tilde{p}_1 r_1^2}$	3.31
Other	First	Odd	Odd	$\frac{p_2 \tilde{p}_0 \tilde{p}_1}{p_1 r_1^2}$	3.76

We can see that, in all cases, the attacker is better off if the defender shifts armies to the front of the line, and in some cases she will be significantly better off, suggesting that the defender should choose to move pairs of armies to the final territory in the chain whenever possible.

Defending fewer territories In situations where the attacker has more than $m + 2$ armies and therefore can lose more than a single army during the campaign,

an enumeration of cases becomes significantly more complicated. However, it is clear that those cases will be a combination of the possibilities enumerated above for situations where the attacker only loses one army at any given time and situations where the attacker loses a pair of armies at the same time, which will affect the probabilities by giving advantages to the attacker of $\frac{p_2}{q_2}$, $\frac{p_2}{r_1^2}$, or $\frac{q_2}{r_1^2}$, all of which are bigger than one. Therefore, we conclude that the defender should always move armies in pairs to the final territory if possible. More specifically, we get the following theorem.

Theorem 3. *In order to best survive an attack on a chain of m territories with n armies, the best defense will be to place either one or two armies on each of the first $m - 1$ territories and the remaining armies on the last territory.*

We have seen that in the case where the attacker has only $m + 1$ armies that it will be best for the defender to place a single army on each of the first $m - 1$ territories. In the following sections, we will see that the opposite conclusion holds if the attacker has an overwhelmingly large number of armies.

Holding off a large number of attacking armies

This section considers the situation in which the attacker has an overwhelmingly large number of attacking armies. In particular, we assume that the defender has a negligible chance of defeating the attacker and instead ask what distribution of the defending armies will do the most damage to the attacker during the campaign. This question is clearly different from our initial question, but we will see that the answer should be similar to the optimal distribution in that case.

Defending a single territory The first case we wish to consider is when the defending player is trying to defend a single territory with d armies on it. Let $E(d)$ be the expected change that a player with a large number of armies will have when attacking a territory with d armies. We note that $E(d)$ will be negative.

Lemma 4. $E(0) = -1$.

In particular, to take over an empty territory, the attacker needs to move a single army onto that territory, depleting her ranks by one.

Lemma 5. $E(1) = \frac{-1}{\tilde{p}_1}$.

Proof. If the defending player has a single army, then on the first round there will be two possibilities: With probability \tilde{p}_1 , the attacker will win and therefore only “lose” the army she needs to use to take over the territory, and with probability p_0 the attacker will lose an army and have to face the defender in the same situation once again. Thus, $E(1) = -\tilde{p}_1 + \tilde{p}_0(-1 + E(1))$. The lemma follows from a simple calculation, noting that $\tilde{p}_1 + \tilde{p}_0 = 1$. ■

When facing more than one army, there are three possible outcomes of a given attack. Considering these three cases, one can see that for $d \geq 2$ we have

$$E(d) = p_2 E(d - 2) + p_1 (E(d - 1) - 1) + p_0 (E(d) - 2),$$

which simplifies to give the following recursive formula for $E(d)$:

$$E(d) = \frac{p_2}{1 - p_0} E(d - 2) + \frac{p_1}{1 - p_0} E(d - 1) - \frac{p_1 + 2p_0}{1 - p_0}.$$

There are several ways to approach nonhomogeneous recurrence relations such as this one; we proceed by noting that one can also see that

$$E(d+1) = \frac{p_2}{1-p_0}E(d-1) + \frac{p_1}{1-p_0}E(d) - \frac{p_1+2p_0}{1-p_0}.$$

Subtracting these two formulae from each other and solving for $E(d+1)$, one obtains the following homogeneous recursion formula for $E(d+1)$:

$$E(d+1) = \left(1 + \frac{p_1}{1-p_0}\right)E(d) + \left(\frac{p_2-p_1}{1-p_0}\right)E(d-1) - \frac{p_2}{1-p_0}E(d-2).$$

The characteristic equation of this recurrence relation is $x^3 - \left(1 + \frac{p_1}{1-p_0}\right)x^2 - \left(\frac{p_2-p_1}{1-p_0}\right)x + \frac{p_2}{1-p_0}$, which factors as $(x-1)^2\left(x + \frac{p_2}{1-p_0}\right)$. It follows from standard results in recurrence relations (see, for example, [3, Ch. 6]) that the generic solution to the recurrence relation takes the form

$$E(d) = c_1 + c_2d + c_3\left(\frac{-p_2}{1-p_0}\right)^d.$$

Using Osborne's values for the p_i , it is easy to compute values for $E(2)$. Using the values of $E(0)$, $E(1)$, and $E(2)$, one can solve for the c_i and obtain the following result.

Theorem 6. *In a given battle in which the defending player has d armies and the attacking player starts with a large enough number of armies that she rolls three dice throughout the battle, the expected number of armies the attacker will lose before taking over the territory is $E(d) = c_1 + c_2d + c_3\alpha^d$ where we define the constants $c_1 = -\frac{578702951}{743204855}$, $c_2 = -\frac{2387}{2797}$, $c_3 = -\frac{164501904}{743204855}$, and $\alpha = -\frac{2890}{5501}$.*

We note that our formula differs somewhat from the formula obtained in [1], in which they state the results as $E(n) = c_1 + c_2n + c_3\alpha^n$ with $c_1 = 0.22134$, $c_2 = -0.85341$, $c_3 = -0.22134$ (and the same value of α). The difference arises because we are including the army that the attacker must "leave behind" when she moves onto the new territory.

Defending a chain of two territories We next consider the case where the defending player has two territories that he wishes to defend. Moreover, he can split the d armies between these two territories, although according to the rules of *RISK* he cannot vacate either territory. In particular, he must choose k with $1 \leq k \leq d-1$ and place k armies on the first territory and $d-k$ armies on the second territory. If he wishes to do this to maximize damage to his opponent, then he is trying to choose k to minimize the function $F(k) = E(k) + E(d-k)$, where E is the function defined in the previous section. In particular, one can use Theorem 6 to compute that $F(k) = 2c_1 + c_2d + c_3(\alpha^k + \alpha^{d-k})$ where α and the c_i are the constants given in the statement of Theorem 6. In particular, we note that the first two terms in this formula are constants with respect to k and moreover that c_3 is negative, so it will suffice for the defending player to maximize the function $\hat{F}(k) = \alpha^k + \alpha^{d-k}$.

We note that the function F is symmetric in the sense that the attacker's expected losses will be the same if the defender places k armies on the first territory and $n-k$ armies on the second territory or the other way around. This is slightly different from actual gameplay in *RISK*, as our computations show that it is actually advantageous to place the smaller number of armies on the first territory and the larger number on

the second territory. To understand why this is so, consider that, in our Markov chain model, the attacker that captures the first territory must leave behind a single army when occupying the captured territory. Thus, the attacker is effectively weakened for the second territory attack, no matter how well the first territory attack proceeds. If defender armies are unevenly distributed between the two spaces, the defender would be more likely to survive if the bulk of the defender armies were met by a weakened attacker. Although in cases with large numbers of armies, this weakening of armies left behind can be subtle, it is nonetheless a measurable advantage. At the end of this article, we will discuss further reasons why our model does not capture actual gameplay with complete accuracy.

Lemma 7. *If $\beta > 0$ and $m > 0$, then the function $\hat{G}(x) = \beta^x + \beta^{m-x}$ is minimized at $x = \frac{m}{2}$ and maximized at both $x = 0$ or $x = m$.*

Proof. The proof of this lemma is a straightforward calculus exercise, as one notes that $\hat{G}'(x) = \ln(\beta)(\beta^x - \beta^{m-x})$. If $0 < \beta < 1$, then $\ln(\beta) < 0$ and $\beta^x - \beta^{m-x}$ will be positive exactly when $x < m - x$. If $\beta > 1$, then both of these signs will be reversed. In either case, \hat{G} will be decreasing for values of $x < \frac{m}{2}$ and increasing for values of x that are greater than $\frac{m}{2}$. ■

Our function is more interesting, as the value of α that we are working with is negative and therefore the function \hat{F} is only defined at integer values of k . To maximize this function, we wish to consider two cases based on the parity of d .

We first consider the case where d is even. Note that for all even values of k the function $\hat{F}(k) = \alpha^k + \alpha^{d-k} = (\alpha^2)^{k/2} + (\alpha^2)^{d/2-k/2}$. Therefore, if k is an even integer, then the function $\hat{F}(k)$ agrees with the function $\hat{G}(k/2)$ as defined in the previous lemma where $\beta = \alpha^2$ and $m = d/2$. This function is maximized by choosing the smallest or largest values of k possible, which in this case must be $k = 2$ or $k = d - 2$ given the restrictions on k .

On the other hand, if k is odd, then we note that $\hat{F}(k) = \alpha(\alpha^{k-1} + \alpha^{d-k-1}) = \alpha((\alpha^2)^{(k-1)/2} + (\alpha^2)^{(d-k-1)/2})$. Because α is negative, we note that this number will always be negative. In particular, it will always be less than $\hat{F}(k)$ for any even choice of k . In particular, we can conclude that if d is even, then $\hat{F}(k)$ is maximized when $k = 2$ (or $k = d - 2$ due to symmetry).

If d is odd, then we begin by noting that for even values of k that the function $\hat{F}(k)$ is equal to $(\alpha^2)^{k/2} + \alpha \cdot (\alpha^2)^{(d-k-1)/2}$. Because $0 < \alpha^2 < 1$, one can compute that this is a continuous function whose derivative is always negative, and therefore the function is strictly decreasing. One can similarly show that $\hat{F}(k)$ is strictly increasing for odd values of k . The fact that $\hat{F}(1) < 0$ and $\hat{F}(2) > 0$ then implies that, for integers between 1 and $d - 1$, this function is maximized at $k = 2$ (or $k = d - 2$ due to symmetry).

We have therefore proved the following theorem.

Theorem 8. *In a situation in which the defending player is trying to defend two consecutive territories with d armies and the attacking army has significantly more than d armies, then the defending player will cause the most damage if he places two armies on one territory and $d - 2$ armies on the other territory.*

It is interesting to note that while the best strategy is to place two armies on one territory and $d - 2$ on the other, the *worst* strategy is actually to divide your armies with one army on one territory and $d - 1$ armies on the other territory and that in general one wishes to place an even number of armies on each territory.

Defending longer chains In this section, we wish to show that the optimal defense for defending a chain of m territories, assuming that one has $d \geq 2m$ armies, will be to place two armies on all but one of the territories and $d - 2(m - 1)$ on the remaining territory. Having shown this in the case where $m = 2$, we now wish to proceed by induction.

Given an arrangement (d_1, \dots, d_m) of armies on the m territories, we define the expected loss of armies by the attacker by the function

$$\begin{aligned} F_m(d_1, \dots, d_m) &= E(d_1) + \dots + E(d_m) \\ &= mc_1 + dc_2 + c_3(\alpha^{d_1} + \dots + \alpha^{d_m}) \end{aligned}$$

where α, c_1, c_2, c_3 are as defined in Theorem 6. In particular, this function will be minimized exactly when the function $\hat{F}_m(d_1, \dots, d_m) = \alpha^{d_1} + \dots + \alpha^{d_m}$ is maximized and the defender's goal is to choose the constants d_1, \dots, d_m with $1 \leq d_i \leq d$ and $\sum d_i = d$ so that $\hat{F}_m(d_1, \dots, d_m)$ is maximized.

We note that for any fixed choice of d_m , we have that $\hat{F}_m(d_1, \dots, d_m) = \alpha^{d_m} + \hat{F}_{m-1}(d_1, \dots, d_{m-1})$. Therefore, this function will be maximized when $\hat{F}_{m-1}(d_1, \dots, d_{m-1})$ is maximized. However, this describes the situation in which a defender is trying to defend $m - 1$ territories with $d - d_m$ armies, and by the inductive hypothesis we know that this will be maximized when all but one of the entries is equal to 2.

Another way of seeing this is by contradiction. Assume that $\hat{F}_m(a_1, \dots, a_m)$ is a maximum value of \hat{F}_m over all m -tuples with $\sum a_i = n$ and further assume that more than one of the a_i is not equal to 2: Without loss of generality, let us assume that $a_1 \neq 2$ and $a_2 \neq 2$. Then $\hat{F}_{m-1}(a_1, \dots, a_{m-1})$ will be a maximum for \hat{F}_{m-1} given the restriction that $\sum a_i = n - a_m$. However, this contradicts the inductive hypothesis, which states that \hat{F}_{m-1} will be maximized when all but one of the entries is equal to two.

We note that the proof of the above theorem is actually quite general and in fact is quite robust in the values of the p_i . In particular, we note that one can conclude that the same strategy is the best if the attacking player always rolls exactly two dice. Given that the actual play in the game of *RISK* is a linear combination of these two scenarios, it seems natural that the best possible strategy to cause the most damage to your opponent in the actual game will be to place two armies on each but the final territory due to a convexity type of argument.

Concluding thoughts

Our model suggests, and proves under various hypotheses, that the best strategy a defending player has in order to fend off an attacker is to place two armies on each territory except the last territory, where he will place the remaining armies. However, there are several aspects of the game of *RISK* that our model does not accommodate for, and we conclude this paper by briefly discussing some of these considerations.

In practice, one will usually reverse the distribution of defending armies, putting the main force at the front. The reason for this is that while this leads to a slight decrease in survival probability, it gives the defending player an increased expectation of retained territories leading to an increased expectation in earned army reinforcements. In the game of *RISK*, a player receives army reinforcements at the beginning of each turn according to the total number of territories the player occupies modulo 3 (with a three army reinforcement minimum) plus bonuses for complete continents occupied.

Thus, while survival is important, a small tradeoff of the probability of immediate survival is advisable to increase the expectation of territory occupation and thus

army reinforcements and thus the probability of longer-term survival. Such long-term strategic considerations are beyond the scope of this tactical paper and would indeed be interesting future work. However, our work has given substantial evidence that the two-army-per-territory defense is the most efficient in certain situations and provided further insight to extreme cases.

Acknowledgment We would like to thank Clifton Presser for his preparation of the graphics used in this paper.

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Summary. We consider a one-dimensional version of the board game *RISK* and discuss the problem of how a defending player might choose to distribute his armies along a chain of territories in order to maximize the probability of survival. In particular, we analyze a Markov chain model of this situation and run computer simulations in order to make conjectures as to the optimal strategies. The latter sections of the paper analyze this strategy rigorously and use results on recurrence relations and probability theory in order to prove a related result.

DARREN GLASS (MR Author ID: [699957](#)) received his B.A. in mathematics and economics at Rice University and his Ph.D. at the University of Pennsylvania, where he worked on questions in arithmetic geometry and Galois theory. Since joining the math department at Gettysburg College in 2005, his research interests have expanded to include number theory, graph theory, sabermetrics, cryptography, and whatever interesting questions colleagues wander into his office to ask. This paper is evidence of the latter.

TODD W. NELLER (MR Author ID: [750401](#)) received his computer science Ph.D. at Stanford University with distinction in teaching in 2000, where he was awarded a Stanford University Lieberman Fellowship, and the George E. Forsythe Memorial Award for excellence in teaching. His dissertation concerned simulation-based search for hybrid system control and analysis. A game enthusiast, Neller has since enjoyed a variety of artificial intelligence game and puzzle challenges, including the computation of optimal play for dice games (e.g., Pass the Pigs, Dudo) and the computer-aided design of logic mazes.



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