Optimal Play of the Great Rolled Ones Game

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Overview

- Rules
- Optimality Equations
- Solution Method
- Optimal Compensation Points and Visualization
- Human-Playable Policies
  - Roll with 4 or 5 dice
  - Fixed hold-at
  - Simple player and ones cases
  - Keep pace, end race, by case
- Conclusions

Great Rolled Ones

- Dice game for 2 or more players. (Here we consider 2 player only.)
- First published in 2020 by Sam Mitschke and Randy Scheunemann
  - Similar to the dice game Zombie Dice
  - Both are jeopardy dice games in the Ten Thousand dice game family
- Jeopardy Dice Game – Primary mechanic: Roll/hold decisions where holding *secures* turn progress, whereas rolling *risks* all turn progress for potentially greater turn progress. “Push your luck.”

Great Rolled Ones Rules

- 2 or more players using 5 standard (d6) dice.
- Players will have the same number of turns. A turn consists of a sequence of player dice rolls where rolled 1s are set aside.
- The turn ends when either the player
  - decides to hold (i.e. stop rolling) and score the total number of non-1s rolled, or
  - has rolled three or more 1s, ending the turn with no score change.
- A round consists of each player taking one turn in sequence.
- Any player ending their turn with a goal score of 50 or more causes that to be the last round of the game.
- At the end of the last round, the player with the highest score wins.
- (We assume that a player is constrained to attempt to exceed the score of the current leader in the last round.)

Great Rolled Ones Example Round

Player	Roll	Result (Decision)
1	1, 1, 3, 4, 5	Two 1s set aside, turn total 3 (roll)
1	2, 2, 4	No 1s set aside, turn total 6 (roll)
1	1, 1, 6	Two 1s set aside for a total of four 1s, ≥ three 1s → **turn ends with no score gain**
2	4, 4, 4, 4, 5	No 1s set aside, turn total 5 (roll)
2	4, 4, 4, 5, 5	No 1s set aside, turn total 10 (roll)
3	1, 1, 2, 4, 5	Two 1s set aside, turn total 13 (hold) → **turn ends with a score gain of 13**

Optimality Equations: Probability of Rolling 1s

Nonterminal states are described as the 5-tuple (p, i, j, k, o), where p is the current player number $(1/2)$, i is the current player score, j is the opponent score, k is the turn total, and o is the number of rolled 1s set aside.

Let $P_{\text{new1s}}(d, o)$ denote the probability that o of d dice rolled are 1s $(0 \leq o \leq d \leq 5)$:

$$P_{\text{new1s}}(d, o) = \binom{d}{o} \left(\frac{1}{6}\right)^{o} \left(\frac{5}{6}\right)^{(d-o)}$$

Optimality Equations: Probability of Player 2 Exceeding Player 1's Winning Score

Let $P_{\text{exceed}}(\Delta, o)$ denote the probability that player 2 will exceed player 1's score ≥ 50 where $\Delta = j - (i + k)$ (their score difference) and o is the number of rolled 1s set aside on player 2's final turn. Then,

$$P_{\text{exceed}}(\Delta, o) = \begin{cases} 0 & \text{if } o \geq 3 \\ 1 & \text{if } \Delta < 0 \\ \sum_{n=0}^{2-o} P_{\text{new1s}}(5 - o, n) P_{\text{exceed}}(\Delta - (5 - o'), o') & \text{otherwise} \\ \qquad \text{where } o' = o + n & \end{cases}$$

Optimality Equations: Probability of Winning with a Roll

The probability of winning with a roll $P_{\mathrm{roll}}(p,i,j,k,o)$ under the assumption of optimal play thereafter is:

$$P_{\mathrm{roll}}(p,i,j,k,o) = \begin{cases} P_{\mathrm{exceed}}(o, j-i) & \text{if } p = 2 \text{ and } j \geq 50 \\ \sum_{n=0}^{2-o} P_{\mathrm{new1s}}(5-o,n) P(p,i,j,k+5-o',o') + \\ \sum_{n=3-o}^{5-o} P_{\mathrm{new1s}}(5-o,n)(1 - P(3-p,j,i,0,0)) & \text{otherwise} \end{cases}$$

A player can (and should) never hold at the beginning of the turn when the turn total is 0, so we express this by treating such rule-breaking as a loss. Thus, ...

Optimality Equations: Probability of Winning with a Hold, Roll/Hold Decision

the probability of winning with a hold $P_{\text{hold}}(p, i, j, k, o)$ under the assumption of optimal play thereafter is:

$$P_{\text{hold}}(p, i, j, k, o) = \begin{cases} 0 & \text{if } k = 0 \text{ or } (p = 2 \text{ and } j \geq 50, i) \\ 1 & \text{if } p = 2 \text{ and } i + k \geq 50 \\ 1 - P(3 - p, j, i + k, 0, 0) & \text{otherwise} \end{cases}$$

Then the probability of winning $P(p, i, j, k, o)$ under the assumption of optimal play is:

$$P(p, i, j, k, o) = \max(P_{\text{roll}}(p, i, j, k, o), P_{\text{hold}}(p, i, j, k, o))$$

…

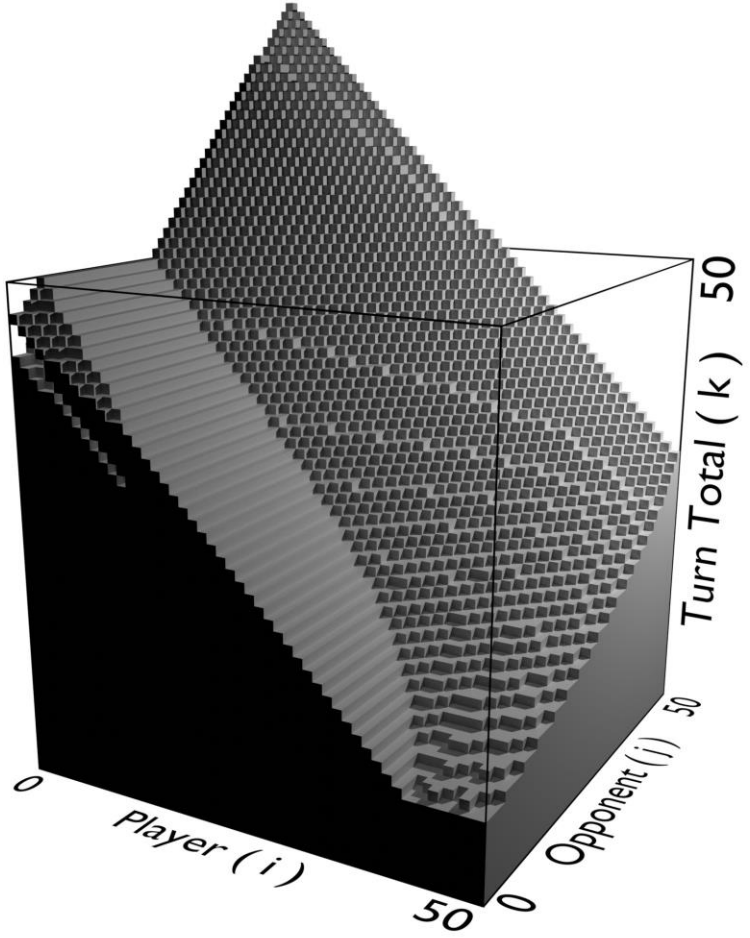
First Player Advantage and Compensation Points (Komi)

- Player 1 finishes with ≥ 50 → Player 2 must exceed Player 1's score
- Player 2 has a knowledge advantage, knowing what score is needed to win.
- With optimal play, player 1 and player 2 have win rates of 0.4495 and 0.5505, respectively (a 10% gap!).
- In the game of Go, "komi" are compensation points designed to make games more fair.
- In the Great Rolled Ones game, player 1 should start with 3 compensation points (komi), bringing player 1's win rate up to 0.4955 (a 0.9% gap) for most fair play.

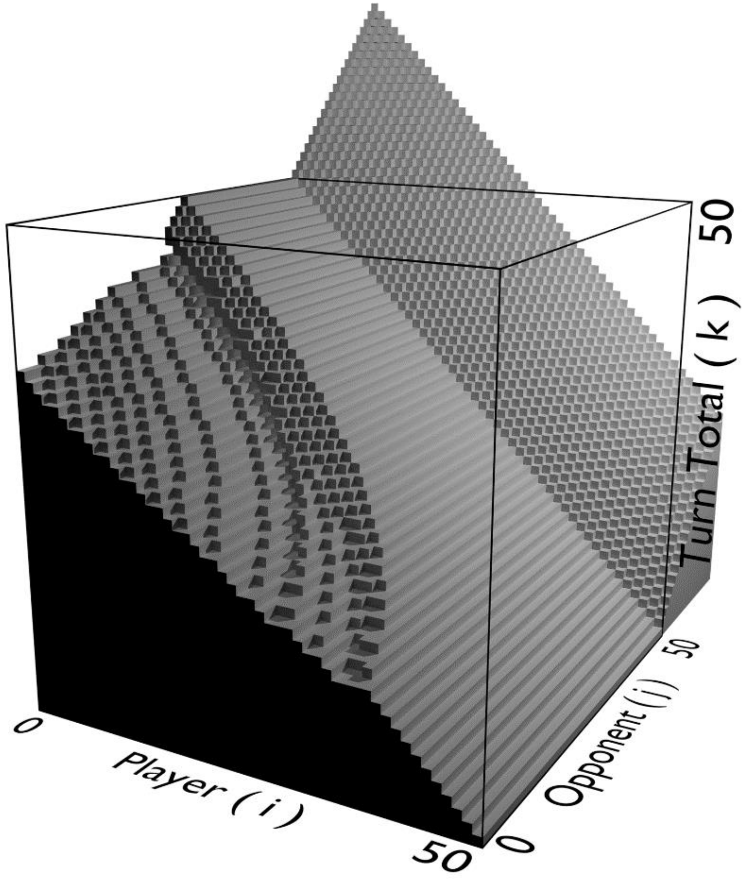
Solving Optimality Equations

- Equations (P_{new1s}, P_{exceed}) are solved through dynamic programming first.
- Cyclic, recursive P is solved through a variation of Value Iteration:
  - From initial arbitrary P estimates, substitute estimates in equation right-hand sides.
  - Compute the left-hand side P values as new, better estimates.
  - Terminate iterations of previous steps when the maximum change to a P estimate is $\leq 1 \times 10^{-14}$.

Optimal Play (zero 1s set aside)

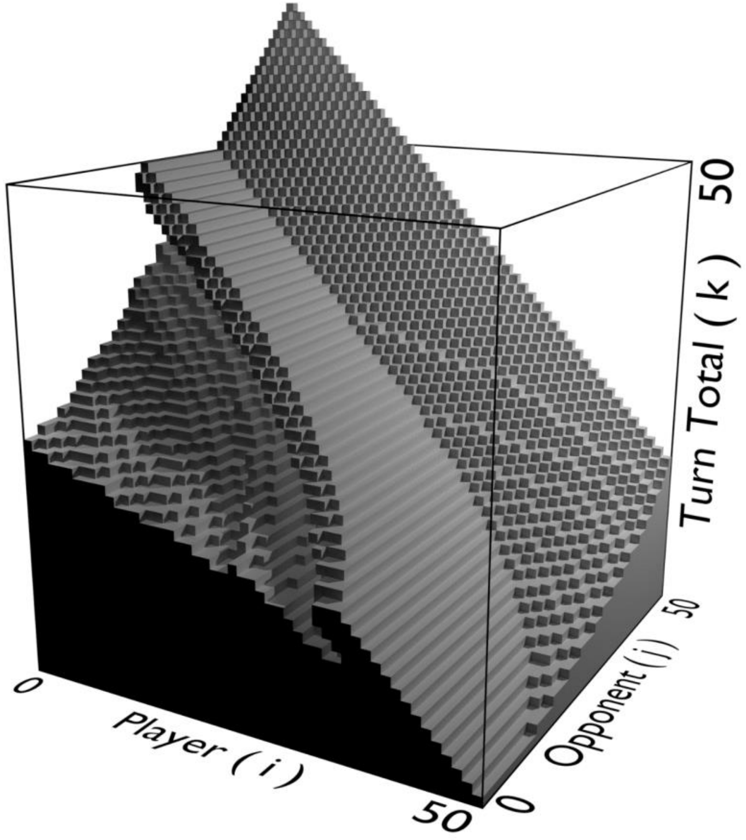


Player 1

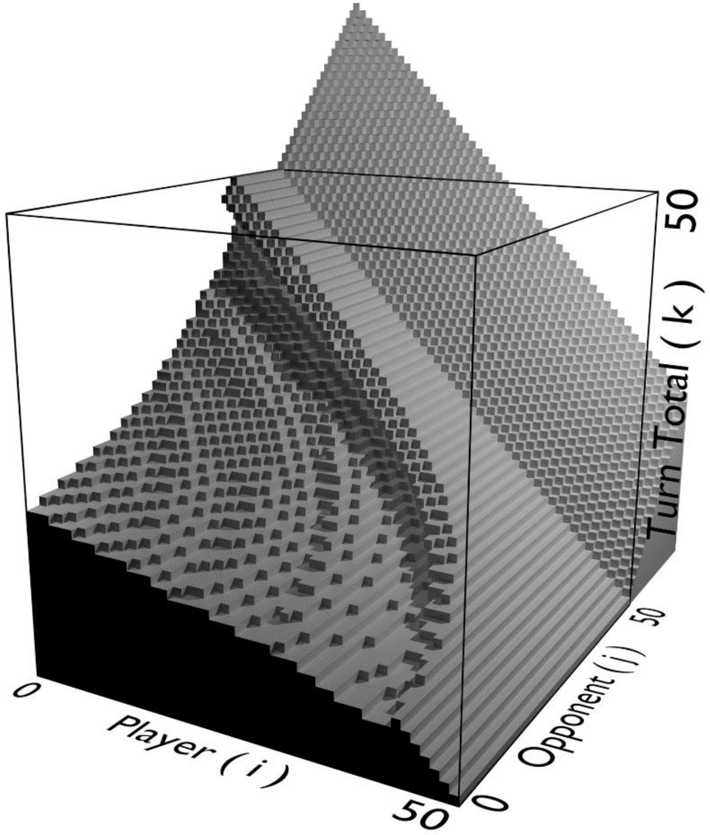


Player 2

Optimal Play (one 1 set aside)

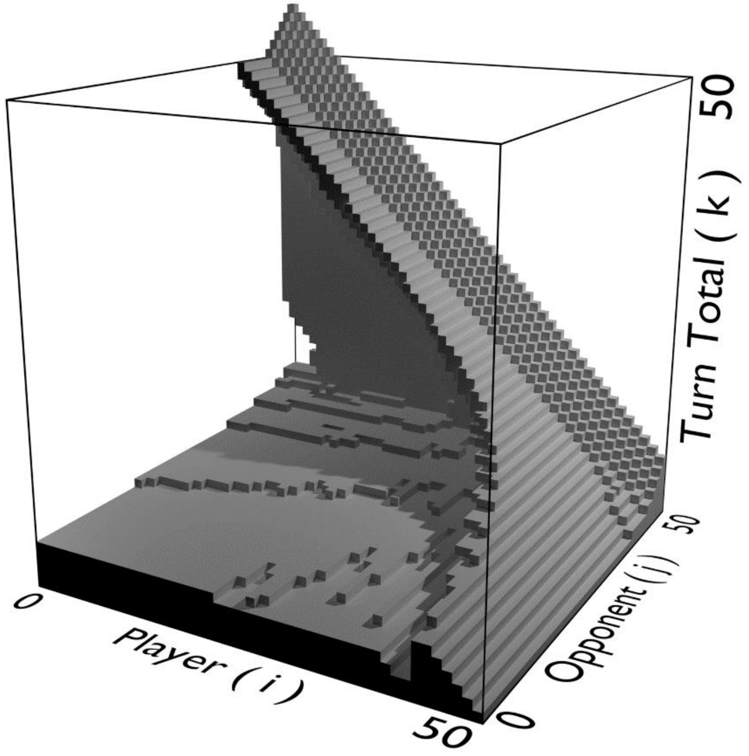


Player 1

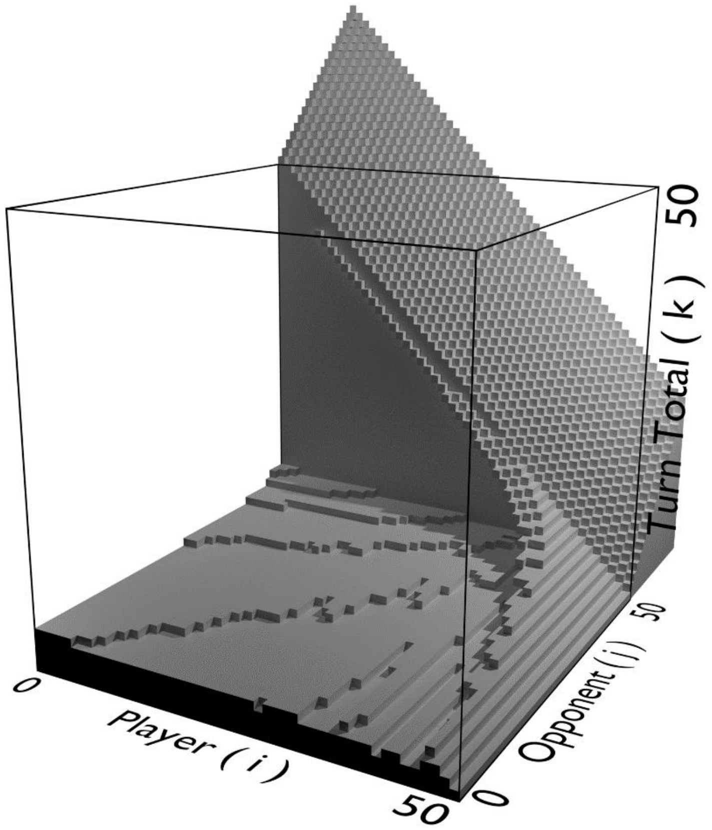


Player 2

Optimal Play (two 1s set aside)



Player 1



Player 2

Human-Playable Policies

- By human-playable, we mean involving simple mental arithmetic and limited recall of cases and constants.
- We will see a continuum of tradeoffs from simple play policies with modest performance, to complex play policies with excellent performance.

Policy	Difference
Roll with 4 or 5 Dice	-0.0536
Fixed Hold-At	-0.0268
Simple Player and Ones Cases	-0.0201
Keep Pace, End Race, by Case	-0.0100

Fig. 2: Differences between human-playable and optimal policy win rates

Roll with 4 or 5 Dice

Algorithm 1: Roll with 4 or 5 dice

Input : player p, player score i, opponent score j, turn total k, ones rolled o

Output: whether or not to roll

if $p = 2 \wedge j \geq 50 \wedge i + k \leq j$ **then** `// player 2 must exceed player 1`

    return *true*

else if $p = 2 \wedge i + k \geq 50$ **then** `// player 2 must hold at goal score`

    return *false*

else `// roll with 4 or 5 dice`

    return $o < 2$

end if

5.4% gap

Fixed Hold-At

Algorithm 2: Fixed hold-at

Input : player p, player score i, opponent score j, turn total k, ones rolled o
Output: whether or not to roll

if $p = 2 \wedge j \geq 50 \wedge i + k \leq j$ **then** `// player 2 must exceed player 1`
  return *true*
else if $i + k \geq 50$ **then** `// player 2 holds and wins`
  return *false*
else if $o = 0$ **then** `// keep rolling with 5 dice`
  return *true*
else if $o = 1$ **then** `// hold at 24 with 4 dice`
  return $k < 24$
else `// hold at 4 with 3 dice`
  return $k < 4$
end if

2.7% gap

Simple Player and Ones Cases

Algorithm 3: Simple player and ones cases

Input : player p, player score i, opponent score j, turn total k, ones rolled o

Output: whether or not to roll

```
if p = 1 then                                          // player 1 cases
    if o = 0 then                                      // keep rolling with 5 dice
        return true
    else if o = 1 then             // hold at goal with ≥ 20 lead with 4 dice
        return k < max(50 − i, 20 + j − i)
    else                                       // hold at 5 or goal with 3 dice
        return k < min(50 − i, 5)
    end if
else                                                   // player 2 cases
    if j ≥ 50 then                             // player 2 must exceed player 1
        return i + k ≤ j
    else if i + k ≥ 50 then                            // hold at goal score
        return false
    else if o < 2 then                                 // roll with 4 or 5 dice
        return true
    else                                       // hold at 5 or goal with 3 dice
        return k < min(50 − i, 5)
    end if
end if
```

2.0% gap

Simple Player and Ones Cases (cont.)

- For player 1, roll with 5 dice. With 4 dice, hold at or beyond the goal with a lead of at least 20. For player 2, if player 1 has reached the goal score, exceed it. Otherwise, if player 2 can hold and win, do so. Otherwise, player 2 always keeps rolling with 4 or 5 dice. With 3 dice, both players should hold if it reaches the goal score or if the turn total is at least 5.

Keep Pace, End Race, by Case

Algorithm 4: Keep pace, end race, by case

Input : player p, player score i, opponent score j, turn total k, ones rolled o

Output: whether or not to roll

```
δ ← j − i
if p = 1 then                                         // player 1 cases
    if o = 0 then              // hold at goal with ≥ 38 lead with 5 dice
        return k < max(50 − i, 38 + δ)
    else if o = 1 then
        h ← 22 + δ                     // hold with a ≥ 22 lead with 4 dice
        if i ≥ 10 ∨ j ≥ 23 then
            // if player 1 / 2 has scored 10 / 23, resp.
            h ← max(50 − i, h)           // then at least roll for the goal
        end if
        return k < h
    else if i + j ≥ 71 then
        // reach the goal when the player score sum reaches 71
        return k < 50 − i
    else                                // hold at 5 or goal with 3 dice
        return k < min(50 − i, 5)
    end if
else                                                  // player 2 cases
    if j ≥ 50 then                      // player 2 must exceed player 1
        return k ≤ δ
    else if o = 0 then                         // keep rolling with 5 dice
        return true
    else if o = 1 then                                    // with 4 dice
        if i ≥ 20 ∨ j ≥ 32 then
            // if player 1 / 2 has scored 20 / 32, resp.
            return k < 50 − i                    // then roll for the goal
        else                                     // else hold with ≥ 28 lead
            return k < 18 + δ
        end if
    else if i + j ≥ 84 then
        // reach the goal when the player score sum reaches 84
        return k < 50 − i
    else                                // hold at 5 or goal with 3 dice
        return k < min(50 − i, 5)
    end if
end if
```

1.0% gap

Future Work

- Supervised learning of win probabilities for nonterminal states could compress our precise tabular computation.
- One-step backup of approximate win probabilities would likely yield excellent roll/hold decisions.
- Question: How well would different models/techniques perform for trading off performance for reduced memory requirements?

Conclusions

- Optimal play has been computed for the Great Rolled Ones game.
- 3 compensation points should be given initially to Player 1 for greatest fairness
- A variety of human playable strategies were presented, including the "Simple Player and Ones Cases" strategy that has a 2% gap from the optimal win rate:
  - (Player 2 must exceed a winning Player 1 score.)

1s Rolled	Player 1	Player2
0	Roll and **do not hold**	Roll to win
1	Hold at ≥ 50 points with a lead of ≥ 20	Roll to win
2	Hold at ≥ 5 turn total or ≥ 50 points	Hold at ≥ 5 turn total or ≥ 50 points